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IDEALS OF COADJOINT ORBITS OF NILPOTENT LIE ALGEBRAS

BY

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ABSTRACT. For f a linear functional on a nilpotent Lie algebra g over a field of characteristic 0, let $J(f)$ be the ideal of all polynomials in $S(g)$ vanishing on the coadjoint orbit through f in g^*, and let $I(f)$ be the primitive ideal of Dixmier in the universal enveloping algebra $U(g)$, corresponding to the orbit. An inductive method is given for computing generators $P_1, \ldots, P_r$ of $J(f)$ such that $\varphi P_1, \ldots, \varphi P_r$ generate $I(f)$, φ being the symmetrization map from $S(g)$ to $U(g)$. Upper bounds are given for the number of variables in the polynomials P_i and a counterexample is produced for upper bounds proposed by Kirillov.

Introduction. Let G be a Lie group, g its Lie algebra. G acts on the dual space g^* by the coadjoint (or contragredient) action. Thus for $f \in g^*$ and $\sigma \in \exp \operatorname{ad} g$, $\sigma(f) = f \circ \sigma^{-1}$. The orbits in g^* under this coadjoint action play an important role in the theory of unitary representations of nilpotent Lie groups and the theory of linear representations of nilpotent Lie algebras.

In his fundamental paper [6] A. A. Kirillov shows that the irreducible unitary representations of a simply connected Lie group are in 1-1 correspondence with the orbits in g^*, and that each such representation is induced by a one-dimensional representation. The principal method is an induction on dimension. If the kernel of the representation contains a normal subgroup of dimension $\geqslant 1$, one passes to the quotient group. If not, then the kernel is discrete and the centre of the group is one dimensional, and one constructs a subgroup of codimension 1 and a representation on the subgroup which induces (in the sense of Mackey) the representation on the original group. Each irreducible representation is induced by a one-dimensional representation corresponding to a point f in g^* and a polarization for f (see 3.1). Two such representations are equivalent iff the points in g^* belong to the same orbit. This gives the correspondence between orbits and irreducible representations. The orbits also give the decomposition of representations into irreducible constituents.

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Dixmier has shown (see [2], [4] or [8]) that there is an analogous situation for linear representations of a nilpotent Lie algebra g and the corresponding linear representations of the universal enveloping algebra $U(g)$. The kernel of an irreducible such representation is called a primitive ideal of $U(g)$. An ideal I of $U(g)$ is rational if $U(g)/I$ is a Weyl algebra. If the ground field is algebraically closed then an ideal is primitive iff it is maximal iff it is rational [2]. The linear representations of g are thus obtained from the primitive ideals of $U(g)$ and the representations of the Weyl algebras.

Dixmier proved that for each point f in g^* there is a rational ideal $I(f)$ in $U(g)$ and that $I(f)$ depends only on the orbit through f, in fact that the orbits correspond to the rational ideals in $U(g)$. Nouazé and Gabriel introduced the notion of rational ideal in the symmetric algebra $S(g)$ (see [8] or [2, 4.8.6]) and proved that the rational ideals of $S(g)$ are exactly the ideals $J(f)$ of all polynomials vanishing on the orbit through f. Thus there is a 1-1 correspondence between the rational ideals of $U(g)$ and those of $S(g)$, via the orbits.

In this paper we shall dèscribe a method of computing a minimal set of generators $P_1, \ldots, P_r$ of $J(f)$ such that $\varphi P_1, \ldots, \varphi P_r$ generate $I(f)$, φ being the symmetrization map from $S(g)$ to $U(g)$. By abuse of language let us call the P_i joint generators of $J(f)$ and $I(f)$. It is surprising that joint generators exist, since $\varphi J(f)$ is not, in general, contained in $I(f)$, as shown by an example of Conze (§10). We illustrate the method of computing joint generators with the low dimensional examples in §§9 and 10.

The proofs use the induction of Kirillov on the dimension of g. If there is an ideal L of g in the kernel of f we say that f is reducible and we pass to the quotient algebra g/L. The induction step is trivial for f reducible. If there is no such ideal L (f is irreducible) then the centre of g is one dimensional and there are elements x,y,z such that $[x,y] = z$, $\langle z \rangle =$ centre g, $f(x) = f(y) = 0$, $f(z) = 1$ and $B =$ kernel ad y is a subalgebra of g of codimension 1. We let f_1 be the restriction of f to B and show that applying the operator $\Gamma = \sum_{j=0}^{\infty}(-y)^j(\text{ad } x)^j/j!$ to joint generators of $J(f_1)$ and $I(f_1)$ gives joint generators of $J(f)$ and $I(f)$.

We need in the proof that $J(f_1) = \langle y, h_1 - R_1, \ldots, h_r - R_r \rangle$, with each R_i a polynomial with no h_j terms (results of this form are well known). In studying $\Gamma(h_i - R_i)$ we need to ensure that no h_i terms arise in ΓR_i, so we introduce a filtration on g and show essentially that R_i can be taken to be a polynomial in elements of (filtered) order smaller than the order of h_i. This ensures that after applying Γ we may make substitutions and obtain a similar expression for $J(f)$. R_i is shown to be a polynomial in fewer than $2k$ variables, where $r + 2k = \dim g$, and r is the dimension of the isotropy algebra g^f (see 3.1). This is of some interest, as in [6] Kirillov made a somewhat stronger claim that R_i could be taken as a polynomial in k variables. §9 contains an example showing that this stronger claim is not true.

The results of this paper appeared previously in [5]. We express our sincere thanks to Victor Guillemin, Shlomo Sternberg, Michèle Vergne and Bertram Kostant for their helpful suggestions and encouragement.

0. **Notation.** g is a finite dimensional filtered nilpotent Lie algebra over a field K of characteristic 0. f is a nonzero element of the dual space g^*. $\langle u_1, \ldots, u_n\rangle$ or $\langle u_1, \ldots, u_n\rangle_S$ is the subspace of S spanned by $u_1, \ldots, u_n$ or the ideal of S generated by $u_1, \ldots, u_n$, as appropriate in context.

$$\operatorname{ad} u(v) = [u,v], \qquad \exp \operatorname{ad} u = \sum_{j=0}^{\infty} \frac{(\operatorname{ad} u)^j}{j!},$$

φ is the symmetrizing map or canonical bijection from the symmetric algebra $S(g)$ to the universal enveloping algebra $U(g)$.

$\varphi(u_1u_2 \cdots u_n) = \Sigma u_{\sigma 1}u_{\sigma 2} \cdots u_{\sigma n}/n!$, the sum over all permutations σ of $\{1, \ldots, n\}$. φ is a g-module map (Dixmier [2, 2.4, p. 79]).

1. Filtrations.

1.1. Definition. $(g, \{F_n\})$ is a *filtered Lie algebra* or $\{F_n\}$ is a *filtration on g* if $F_n \subset F_{n+1}$ and $[g, F_{n+1}] \subset F_n$ for each integer n. We define the *order* of a nonzero $u \in g$ by $o(u) = j$ if $u \in F_j$ and $u \notin F_{j-1}$. We define $o(0) = -\infty$. We say u has *minimal order* if u has minimal finite order.

1.2. Lemma.

$$o([u,v]) \leqslant o(v) - 1, \quad o(u + v) \leqslant \sup\{o(u), o(v)\},$$
$$o(u + v) = o(u) \quad \text{if } o(v) < o(u).$$

1.3. Definition. $\{u_i\}$ is a *filtered basis* for $(g,\{F_n\})$ if any element of order n can be expressed as a sum of basis elements of order $\leqslant n$. A filtered basis $\{u_i\}$ for $(g,\{F_n\})$ is *consistent* on a subspace B of g if those basis elements u_i in B form a filtered basis for B with the restricted filtration $\{F_n \cap B\}$. The order function o on $(B,\{F_n \cap B\})$ is the restriction to B of the order function o on $(g,\{F_n\})$.

1.4. Lemma. *A filtered basis $\{u_1, \ldots, u_r\}$ for a subspace $(B,\{F_n \cap B\})$ of $(g, \{F_n\})$ can be extended to a filtered basis for $(g, \{F_n\})$ by: having chosen $u_1, \ldots, u_{m-1}$, choosing u_m to be an element of minimal order in the complement of $\langle u_1, \ldots, u_{m-1}\rangle$.*

1.5. Lemma. *Suppose $\{u_1, \ldots, u_r\}$ is a filtered basis for $(g, \{F_n\})$, consistent on an ideal K, and π is the natural projection to g/K. Then*

(1) *$\{\pi F_n\}$ is a filtration on g/K and $o(\pi u) = \inf\{o(v)\colon \pi v = \pi u\}$.*

(2) *$\{\pi u_i\colon \pi u_i \neq 0\}$ is a filtered basis for $(g/K,\{\pi F_n\})$.*

(3) *Conversely, if* $\{v_i\}$ *and* $\{w_j\}$ *are filtered bases for* $(K,\{F_n \cap K\})$ *and* $(g/K,\{\pi F_n\})$, *respectively, and for each* $j, \pi x_j = w_j$ *and* $o(x_j) = o(w_j)$, *then* $\{v_i\} \cup \{x_j\}$ *is a filtered basis for* $(g,\{F_n\})$.

We may note that x_i is an element of minimal order under the condition that $\pi x_i = w_i$.

1.6. REMARK. The ascending central series and an inversely labeled descending central series are filtrations for a nilpotent Lie algebra. The elements of minimal order are in the centre.

2. The induction: reducible and irreducible functionals. Many of our arguments will be by an induction on the dimension of g used by Kirillov. The induction takes two forms, according to the reducibility or irreducibility of f, in the sense defined below.

2.1. DEFINITION. f is *reducible* if centre(g) $\cap$ kernel(f) is nonempty. If f is reducible, choose u of minimal order in centre $g \cap$ kernel f and let π be the natural map from g to $C = g/\langle u\rangle$, with the function f_2 induced on C by f.

2.2. DEFINITION. f is *irreducible* if centre $g = \langle z\rangle$ with $f(z) = 1$. Then z has minimal order in g. Let y be an element of minimal order in kernel f. Then $[y,g] = \langle z\rangle$ and, hence, B = kernel ad y has codimension 1 in g, and there is an x of minimal order in the complement of B with the additional properties that $[x,y] = z$ and $f(x) = 0$. Let $C = B/\langle y\rangle, f_1 = f$ restricted to $B, f_2 =$ the function induced on C by f_1. Then $o(\pi v) = o(v)$ when $v \not\in \langle y,z\rangle$. If f is irreducible we pass to f_1 and thence to f_2. We shall refer repeatedly to the specific notation of these two constructions.

3. Reductions of isotropy algebras.

3.1. DEFINITION. The *isotropy algebra* g^f is the kernel of the bilinear form $B_f(u,v) = f([u,v])$, that is, $g^f = \{u \in g: f([u,g]) = 0\}$. D is a *polarization* for f in g if D is a subalgebra of g of dimension $(\dim g + \dim g^f)/2$ and $f([D,D]) = 0$. D is a totally isotropic subspace of maximal dimension for the form B_f and, hence, contains g^f.

3.2. We shall discuss in this section the relation between the isotropy algebra g^f and an isotropy algebra for a linear functional induced from f on a Lie algebra of lower dimension. We note that f is reducible iff there is a nonzero ideal of g in the kernel of f.

3.3. LEMMA. ONE-STEP REDUCTION OF g^f: f REDUCIBLE. *Suppose* V *is an ideal of* g *and* $f(V) = 0$. *Let* $A = g/V$, *let* π *be the natural map from* g *to* A, *and let* f_2 *be the map on* A *induced by* f. *Then* $g^f = \pi^{-1}(A^{f_2})$, $\pi(g^f) = A^{f_2}$ *and* D *is a polarization for* f_2 *if* $\pi^{-1}(D)$ *is a polarization for* f.

3.4. LEMMA. TWO-STEP REDUCTION OF g^f: f IRREDUCIBLE. *Suppose f is irreducible, with $(x, y, z, B, f_1, C, f_2)$ as in* 2.2. *Then* $g^f + \langle y \rangle = B^{f_1} = \pi^{-1}(C^{f_2})$ *and π is an isomorphism of g^f with C^{f_2}.*

PROOF. We note that this two-step reduction from g to C has disposed of x and y but "preserved" g^f, whereas the one-step reduction disposed of part of g^f. We shall see from Lemma 7.3 that the two-step reduction involves a distinct loss of information on the orbit through f.

$g^f \subset B$, since $u \notin B$ implies $f([u,y]) \neq 0$. Thus g^f is a hyperplane in B^{f_1} (by 1.12.2, p. 54, in Dixmier [2]) and therefore $B^{f_1} = g^f + \langle y \rangle$. By 3.3, $\pi^{-1}(C^{f_2}) = B^{f_1} = g^f + \langle y \rangle$ and $\pi(g^f) = C^{f_2}$; and thus π is an isomorphism of g^f onto C^{f_2}.

4. **The ideals $J(f)$ and $I(f)$.**

4.1. Let G be the group generated by exp ad u, $u \in g$. Since g is nilpotent, $G = \exp \operatorname{ad} g$. G acts on g^* by the coadjoint action: $\sigma(h)(v) = h(\sigma^{-1}v)$ for $h \in g^*, v \in g, \sigma \in G$.

4.2. $\omega(f)$ is the orbit in g^* of G acting on f.

4.3. $J(f)$ is the set of all polynomials in $S(g)$ vanishing on $\omega(f)$.

4.4. Let D be a polarization (3.1) for f. Let $M(D,f)$ be the left ideal of $U(g)$ generated by the elements $d - f(d)$, $d \in D$. Let $I(D,f)$ be the largest two-sided ideal of $U(g)$ in $M(D,f)$. Dixmier has shown that $I(D,f)$ is independent of D, whence we shall write $I(D,f) = I(f)$, and that $I(f) = I(h)$ if f and h belong to the same orbit in g^*.

4.5. LEMMA. *If u is in the centre of g, then $u - f(u)$ is in $J(f)$ and in $I(f)$.*

4.6. LEMMA. *If $u \in g^f$ then* $\exp \operatorname{ad} u(f) = f$.

5. **Kirillov projections.** Suppose B is a subalgebra of codimension 1 in a nilpotent Lie algebra g. Let ι: $B \to g$ and ι^*: $g^* \to B^*$ be the natural maps, $f_1 = \iota^* f$. Kirillov showed that the orbits $\omega(f)$ behaved in two ways under ι^*: ι^* is a *type* I *projection* on $\omega(f)$ if $\iota^{*-1}\omega(f_1) \cap \omega(f) = \{f\}$. In this case, ι^* is 1-1 on $\omega(f)$, $\iota^*\omega(f) = \omega(f_1)$, and $\iota^{*-1}\omega(f_1)$ is the union of a one parameter family of orbits in g^*, one for each point in $\iota^{*-1}(f_1)$. ι^* is a *type* II *projection* on $\omega(f)$ if $\iota^{*-1}(f_1) \subset \omega(f)$. In this case, $\iota^*\omega(f)$ is the disjoint union of the orbits through $\sigma_t f_1$, where $\sigma_t = \exp \operatorname{ad} tx$, $x \in g - B$, $t \in K$. Type II projections reduce the dimension of the orbit by 2. For every orbit of dimension > 0 there is a subalgebra of codimension 1 containing g^f, and hence there is a type II projection of the orbit. If every subalgebra of codimension 1 contains g^f then there will not be a type I projection.

5.1. THEOREM. *If $g^f \not\subset B$ then ι^* gives a Kirillov type* I *projection of $\omega(f)$ into*

B^*, and $J(\iota^*\omega(f)) = J(f_1) = J(f) \cap S(B)$ (or, more precisely, $\iota^{-1}(J(f) \cap \iota S(B))$).

Since we do not use Theorem 5.1 in the sequel we shall omit the proof.

5.2. THEOREM. *If $g^f \subset B$ then ι^* gives a Kirillov type* II *projection of $\omega(f)$ into B^*, and*

$$J(f) = S(g)\cdot \iota J(\iota^*\omega(f)) \doteq S(g)\cdot \iota \bigcap_{t\in K} J(\sigma_t f_1).$$

That is, the ideal of $\omega(f)$ is the extension to $S(g)$ of the ideal $\iota^*\omega(f)$.

PROOF. We remark that for later theorems x,y and B below may be as in 2.2. Since $B \supset g^f$ and B has codimension 1 in g, g^f is a hyperplane in B^{f_1} and, hence, $B^{f_1} = g^f + \langle y\rangle$ (Dixmier [2, 1.12.2, p. 54]). Since $y \in g^f$ we choose an x such that $f[x,y] = 1$. Let θ be in g^*, $\theta(x) = 1$ and $\theta(B) = 0$. Let $\sigma_t = \exp \operatorname{ad} tx$, $t \in K$, and let $G = \exp \operatorname{ad} g$. Then G is the semidirect product of $\exp \operatorname{ad} B$ with the 1-parameter subgroup $\{\sigma_t\}$.

We complete the proof of 5.2 with Lemmas 5.3 to 5.5.

5.3. LEMMA. $\omega(f) \supset \iota^{*-1}(f_1) = \bigcup_{t\in K}\{\exp \operatorname{ad} ty(f)\}$.

PROOF. Let $w \in g$. Then by 4.1,

$$\begin{aligned}\exp \operatorname{ad} ty(f)(w) &= f(\exp \operatorname{ad} - ty(w))\\ &= f(w) - tf[y,w] + \sum_{n=1}^{\infty}(-t)^{n+1}f[y, \operatorname{ad} y^n(w)]\\ &= f(w) + tf[w,y].\end{aligned}$$

It follows that $\exp \operatorname{ad} ty(f) = f + t\theta$.

5.4. LEMMA. $\omega(f) = \iota^{*-1} \bigcup_{t\in K}\omega(\sigma_t f_1)$.

PROOF. This follows from 5.3. If $\mu \in G$, then $\mu = \sigma\sigma_t$ with $\sigma \in \exp \operatorname{ad} B$ and $\iota^*(\sigma\sigma_t f) = \sigma\iota^*(\sigma_t f)$.

5.5. LEMMA. *Let $L = \bigcap_{t\in K} J(\sigma_t f_1)$, in $S(B)$. Then $J(f) = S(g)\cdot L$ in $S(g)$.*

PROOF. In the statement of 5.5 and hereafter we take $S(B) \subset S(g)$, but it is clearer to prove the more exact statement: $J(f) = S(g)\cdot \iota L$.

(i) If $P \in L$ then $\iota P(\omega(f)) = P(\iota^*\omega(f)) = 0$. Hence $J(f)$ contains ιL.

(ii) Let $P \in J(f)$. Then $P = \sum \iota(a_j)x^j$, $a_j \in S(B)$.

$$\begin{aligned} P(\sigma f + t\theta) &= \sum a_j(\iota^*\sigma f)(x(\sigma f + t\theta))^j \\ &= \sum a_j(\iota^*\sigma f)((\sigma f)(x) + t)^j \\ &= 0 \quad \text{for all } t \in K. \end{aligned}$$

Thus $a_j \in L$ for all j.

This completes the proof of 5.2.

6. Lemmas.

6.0. Let

$$\Gamma = \sum_{j=0}^{\infty} (-y)^j \frac{(\text{ad } x)^j}{j!}, \quad \sigma_t = \sum_{j=0}^{\infty} t^j \frac{(\text{ad } x)^j}{j!}.$$

6.1. LEMMA. *For S in $S(g)$ or $U(g)$*, ad $x(\Gamma S) = -(z-1)\Gamma(\text{ad } x(S))$.

6.2. LEMMA. *If S is in $S(g)$ or $U(g)$ and $T = \Gamma(S)$. Then*

$$\sigma_t(AT) = \sigma_t(A)T + D(z-1), \quad \Gamma(AT) = \Gamma(A)T + E(z-1).$$

PROOF. This follows from 6.1.

6.3. LEMMA. *For S in $S(g)$ or $U(g)$*,

$$\text{ad } x^j(yS) = jz(\text{ad } x^{j-1}S) + y(\text{ad } x^j S).$$

6.4. LEMMA. *For S in $S(g)$ or $U(g)$*,

$$\Gamma(yS) = (z-1)(-y)\Gamma(S).$$

6.5. LEMMA (STANDARD SUBSTITUTION LEMMA). *If $I = \langle a - b, P_1(a, x_2, \ldots, x_n), \ldots, P_j(a, x_2, \ldots, x_n)\rangle$ is an ideal in $S(g)$ or $U(g)$ then $I = \langle a - b, P_1(b, x_2, \ldots, x_n), \ldots, P_j(b, x_2, \ldots, x_n)\rangle$. That is, we can replace a by b if we retain $a - b$ in I.*

6.6. *Localization*. Let $U = U(g)$, z be in the centre of U, and let $U_z = \{u/z^n : u \in U, n \text{ a nonnegative integer}\}$, with $(u/z^n)(v/z^m) = uv/z^{n+m}$ and with $u/z^n = v/z^m$ iff $uz^m = vz^n$ in U. The map ρ, $\rho(u) = u/1$, is an injection of U into U_z and we may use it to identify u and $u/1$. If I is an ideal in U then $I_z = U_z \cdot \rho(I) = \{u/z^n : u \in I\}$ is an ideal in U_z.

6.7. LEMMA. *If I is an ideal of U and $z - 1$ is in I then $I = I_z \cap U$.*

PROOF. (More exactly, $\rho(I) = I_z \cap \rho(U)$.) Clearly $\rho(I) \subset I_z \cap \rho(U)$. But if $u/z^n = w/1$, $u \in I$ and $w \in U$, then $u = wz^n = w(z^n - 1) + w$ in U, and hence w is in I.

6.8. LEMMA. *If I is an ideal in $U(g)$ containing $z - 1$ and $I_z = \langle z - 1, P_1, \ldots, P_m\rangle_{U_z}$ with each P_i in U then $I = \langle z - 1, P_1, \ldots, P_m\rangle_U$.*

The proof is similar to that of 6.7.

6.9. *Notation*. Let A_1 be the Weyl algebra over K with generators p and q, that is, the associative algebra generated by p and q with the relation $pq - qp = 1$.

6.10. LEMMA (DIXMIER [4]). *Suppose* $U = U(g)$ *and* I *is a two-sided ideal in* $A_1q \otimes U + A_1 \otimes M$, *where* M *is a vector subspace of* U. *Then* $I = A_1 \otimes J$, *where* J *is a two-sided ideal of* U *contained in* M. *This result holds also if* I *is in* $A_1p \otimes U + A_1 \otimes M$.

6.11. We now state a result of Nouazé and Gabriel. We assume that f is irreducible, with x,y,z,B,C as in 2.2. We define a homomorphism ψ from $U(g)_z$ to $A_1 \otimes_K U(C)_{\pi z}$, where π is the natural map from B to $C = B/\langle y\rangle$, by $\psi(x) = p \otimes 1$ and $\psi(u) = \sum_{j=0}^{\infty} q^j \otimes \pi((\mathrm{ad}\, x)^j u)/j!$ for u in B.

6.12. LEMMA (NOUAZÉ AND GABRIEL [8, p. 79] OR [2, 4.7.8, p. 150]). *With* ψ *as above*,

(i) ψ *is an algebra isomorphism.*

(ii) $\psi^{-1}(p \otimes 1) = x$.

(iii) $\psi^{-1}(q \otimes 1) = y/z$.

(iv) $\psi^{-1}(1 \otimes u_0) = \sum_{j=0}^{\infty}(-y/z)^j\, \mathrm{ad}\, x^j u/j!$, *where* u *is any element in* $U(g)$ *with* $\pi(u) = u_0$.

6.13. LEMMA (DIXMIER-NOUAZÉ-GABRIEL). *Suppose* f *is irreducible. Then* $\psi(I(f)_z) = A_1 \otimes I(f_2)_{\pi z}$.

PROOF. The lemma follows from 6.10 and 6.12, or see Dixmier [2, 6.2.1, p. 190].

6.14. LEMMA. *Suppose* f *is reducible. Then* (*see* 2.1 *for notation*)

(i) $J(f) = \pi^{-1}J(\bar{f}), \pi J(f) = J(\bar{f})$.

(ii) $J(f) = \pi^{-1}J(f_2), \pi J(f) = J(f_2)$.

(iii) $I(f) = \pi^{-1}I(f_2), \pi I(f) = I(f_2)$.

7. Theorem on generators of $I(f)$ and $J(f)$.

7.1. THEOREM. *Suppose* g *is a filtered nilpotent Lie algebra*, $f \in g^*$. *Then there is a filtered basis* $\{h_1, \ldots, h_r, x_1, \ldots, x_n, y_1, \ldots, y_n\}$ *of* g *such that*:

(1) $g^f = \langle h_1, \ldots, h_r\rangle$.

(2) *There exist* $P_1, \ldots, P_r \in S(g)$ *such that* $J(f) = \langle P_1, \ldots, P_r\rangle_{S(g)}$ *and* $I(f) = \langle \varphi P_1, \ldots, \varphi P_r\rangle_{U(g)}$, *where* φ *is the symmetrization map from* $S(g)$ *to* $U(g)$.

(3) $J(f) = \langle h_1 - R_1, \ldots, h_r - R_r\rangle$, *where each* R_i *is a polynomial in elements of* $\{x_1, \ldots, x_n\}$ *of order* $< o(h_i)$ *and elements of* $\{y_1, \ldots, y_n\}$ *of order* $< o(h_i)$.

PROOF. The theorem follows from 7.2 and 7.3 by induction on dim g.

7.2. LEMMA (INDUCTION STEP FOR f REDUCIBLE). *Suppose f is reducible, with u of minimal order in* kernel $f \cap$ centre g, *π the natural map to $\bar{g} = g/\langle u\rangle$. Let $\{\bar{h}_2, \ldots, \bar{h}_r, \bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_n\}$, $\bar{R}_i, \bar{P}_i$, satisfy Theorem* 7.1 *for $\bar{g}$ with the projected filtration. We obtain a basis and polynomials satisfying* 7.1 *in g by lifting the basis to $\{h_1, \ldots, h_r, x_1, \ldots, x_n, y_1, \ldots, y_n\}$ by* 1.5(3), *with $h_1 = \underline{R}_1 = P_1 = u$ and $R_2, \ldots, R_r, P_2, \ldots, P_r$ any polynomials such that $\pi R_i = \bar{R}_i$, $\pi P_i = \bar{P}_i$.*

PROOF. The lemma follows from 6.14, since $\pi\varphi = \varphi\pi$.

7.3. LEMMA (INDUCTION STEP FOR f IRREDUCIBLE). *Suppose f is irreducible, with x,y,z,B as in* 2.2. *Suppose Theorem* 7.1 *is satisfied in B with the restricted filtration and $\{y,z,h_2, \ldots, h_r, x_2, \ldots, x_n, y_2, \ldots, y_n\}$ is a filtered basis for B such that $B^f = \langle y,z,h_2, \ldots, h_r\rangle$, with $y, z-1, P_2, \ldots, P_r$ polynomials satisfying* 7.1(2) *and $y, z-1, h_2 - T_2, \ldots, h_r - T_r$ polynomials satisfying* 7.1(3), *in B. Then $\{z,h_2, \ldots, h_r, x, x_2, \ldots, x_n, y, y_2, \ldots, y_n\}$ is a filtered basis for g and*

(1) $g^f = \langle z,h_2, \ldots, h_r\rangle$;

(2)

$$J(f) = \langle z-1, \Gamma P_2, \ldots, \Gamma P_r\rangle_{S(g)}$$

and

$$I(f) = \langle z-1, \varphi\Gamma P_2, \ldots, \varphi\Gamma P_r\rangle_{U(g)},$$

with Γ as in 6.0;

(3) $J(f) = \langle z-1, \Gamma(h_2 - T_2), \ldots, \Gamma(h_r - T_r)\rangle$ *and by substitution we obtain $J(f) = \langle z-1, h_2 - R_2, \ldots, h_r - R_r\rangle$ as required in* 7.1(3).

7.4. REMARK. We prove Lemma 7.3 in §8. Lemma 4.5 justifies the presence of y and $z-1$ in $J(f_1)$ and $I(f_1)$.

7.5. REMARK. The procedures outlined in 7.1 to 7.3 are not merely existence proofs but allow us to compute the polynomials $P_1, \ldots, P_r$. We shall indicate how this is done. If $g^f = g$ then $g = \langle h_1, \ldots, h_r\rangle$ and $P_i = h_i - f(h_i)$ for each i. If $g^f \neq g$ we may select a basis $\{z,h_2, \ldots, h_r, x_1, \ldots, x_n, y_1, \ldots, y_n\}$ for g as follows:

(i) If f is reducible, choose a basis element h_i ($i > 2$) of minimal order in centre $g \cap$ kernel f. h_i is in g^f.

(ii) Project to $g/\langle h_i\rangle = \bar{g}$ with the projected filtration.

(iii) Repeat steps (i) and (ii) in the quotient algebra obtained, lifting the basis elements chosen to g by 1.5, preserving the order, until the induced f is irreducible on the quotient algebra. (The quotient algebra will be $g/\langle h_2, \ldots, h_j\rangle$ the first time steps (i) to (iii) are followed.)

(iv) If f is irreducible, choose x_i, y_i, z, B (in the quotient algebra), as in 2.2, again lifting the basis elements to g by 1.5. Take B with the restricted filtration and then project to $B/\langle y\rangle$.

(v) Repeat steps (i) to (iv) until the quotient is a commutative algebra.

This procedure is illustrated in §9. Repeated application of 7.2 and 7.3 with these basis elements will show that the polynomials $P_1 = z - 1$, $P_i = \Gamma_1\Gamma_2 \cdots \Gamma_r(h_i)$, $i = 2, \ldots, r$, satisfy Theorem 7.1(2), where $\Gamma_m = \Sigma_{j=0}^{\infty}(-y_m)^j(\text{ad } x_m)^j/j!$. In fact, by linearity the polynomials $P_i = \Gamma_1\Gamma_2 \cdots \Gamma_r(h_i) - f(h_i)$, $i = 1, \ldots, r$, will satisfy Theorem 7.1(2) if $h_1, \ldots, h_r$ is any basis for g^f and $x_1, \ldots, x_n, y_1, \ldots, y_n$ are chosen as above.

8. **Proof of 7.3.** We use the notation of 6.0 and 7.3. If g is commutative 7.1 and 7.3 follow from 3.1 and 4.5.

8.1. Lemma. $\sigma_t J(f_1) = J(\sigma_t f_1)$.

8.2. Lemma. *If* $J(f_1) = \langle y, z - 1, Q_2, \ldots, Q_r\rangle$ *then* $\sigma_t J(f_1) = \langle y + t, z - 1, \Gamma Q_2, \ldots, \Gamma Q_r\rangle$.

Proof. $\sigma_t(y) = y + tz, \sigma_t(z - 1) = z - 1, \sigma_t(Q_i) = \Sigma t^j \text{ad } x^j(Q_i)/j!$.

Since $t = -y + (y + tz) - t(z - 1)$, we can replace t by $-y$ and z by 1, by 6.5, to get the desired generators.

8.3. Lemma. $\sigma_t J(f_1) = \langle y + t, z - 1, h_2 - R_2, \ldots, h_r - R_r\rangle$ *with* R_j *a polynomial in elements of* $\{x_1, \ldots, x_n\}$ *of order* $< o(h_j)$ *and elements of* $\{y_1, \ldots, y_n\}$ *of order* $\leqslant o(h_j)$.

Proof. $J(f_1) = \langle y, z - 1, \Gamma(h_2 - T_2), \ldots, \Gamma(h_r - T_r)\rangle$ by 8.1 and 8.2. Each T_i is a polynomial in x_j of order $\leqslant o(h_i)$ and

$$\Gamma(h_i - T_i) = h_i - T_i + \sum_{j=1}^{\infty} (-y)^j \text{ad } x^j(h_i - T_i)/j!.$$

Now $o(y) \leqslant o(h_i)$ and $\text{ad } x(h_i - T_i)$ is a polynomial in elements of order $< o(h_i)$. Thus, if h_j appears in the above expression for $\Gamma(h_i - T_i)$ then $o(h_j) < o(h_i)$ and, by induction on $o(h_i)$, $h_j - R_j \in \sigma_t J(f_1)$, where R_j satisfies the conditions of the conclusion. Thus we may substitute R_j for h_j in the expression for $\Gamma(h_i - T_i)$, by 6.5.

8.4. Lemma.

$$\begin{aligned} J(f) &= \langle z - 1, h_2 - R_2, \ldots, h_r - R_r\rangle \\ &= \langle z - 1, \Gamma(h_2 - T_2), \ldots, \Gamma(h_r - T_r)\rangle. \end{aligned}$$

Proof. The R_j are as in 8.3. In $S(B)$,

$$\bigcap_t \sigma_t J(f_1) = \bigcap_t \langle y + t, z - 1, h_2 - R_2, \ldots, h_r - R_r\rangle.$$

Let $A = \langle z - 1, h_2 - R_2, \ldots, h_r - R_r\rangle_{S(B)}$ and let μ: $S(B) \to S(B)/A$ be the natural map. Then

$$S(B)/A \simeq K[x_2, \ldots, x_n, y, y_2, \ldots, y_n]$$

since each R_i is a polynomial in $\{x_2, \ldots, x_n, y_2, \ldots, y_n\}$. But $\mu\sigma_t J(f_1) = \langle y + t\rangle$ in $S(B)/A$ so $\mu \bigcap_t \sigma_t J(f_1) \subset \bigcap_t \langle y + t\rangle = \{0\}$. Hence, $\bigcap_t \sigma_t J(f_1) = A$ and thus by 5.2, $J(f) = \langle z - 1, h_2 - R_2, \ldots, h_r - R_r\rangle_{S(g)}$. But reversing the substitutions for h_j (in the proof of 8.3) gives us

$$J(f) = \langle z - 1, \Gamma(h_2 - T_2), \ldots, \Gamma(h_r - T_r)\rangle_{S(g)}.$$

8.5. Lemma. *If* $J(f_1) = \langle y, z - 1, Q_2, \ldots, Q_s\rangle_{S(B)}$, *then* $J(f) = \langle z - 1, \Gamma Q_2, \ldots, \Gamma Q_s\rangle_{S(g)}$.

Proof.

$$J(f_1) = \langle y, z - 1, \Gamma(h_2 - T_2), \ldots, \Gamma(h_r - T_r)\rangle = \langle y, z - 1, \Gamma Q_2, \ldots, \Gamma Q_s\rangle$$

and thus

$$\Gamma(h_i - T_i) = A_0 y + A_1(z - 1) + \sum A_i \Gamma Q_i.$$

Applying Γ,

$$\Gamma(h_i - T_i) + D(z - 1) = \sum \Gamma(A_i)\Gamma(Q_i) + E(z - 1),$$

by 6.1, 6.4, and, hence, $J(f) \subset \langle z - 1, \Gamma Q_2, \ldots, \Gamma Q_r\rangle$ by 8.4. The reverse inclusion follows from 8.2 and 5.5.

8.6. Lemma. *Suppose* $J(f_1) = \langle y, z - 1, S_2, \ldots, S_r\rangle_{S(B)}$ *and* $I(f_1) = \langle y, z - 1, \varphi S_2, \ldots, \varphi S_r\rangle_{U(B)}$. *Then* $I(f) = \langle z - 1, \Gamma\varphi S_2, \ldots, \Gamma\varphi S_r\rangle_{U(g)}$.

Proof. Let $C = \pi(B) = B/\langle y\rangle$ and f_2 be the map induced on C by f. Then $J(f_2) = \langle \pi z - 1, \pi S_2, \ldots, \pi S_r\rangle_{S(C)}$ by 6.14. Let $T_i = \Gamma S_i$. Then $\pi T_i = \pi S_i$ and $I(f_2) = \langle \pi z - 1, \varphi\pi T_2, \ldots, \varphi\pi T_r\rangle$, since $\varphi\pi = \pi\varphi$. From 6.13, $\psi(I(f)_z) = A_1 \otimes I(f_2)_{\pi z}$. But $I(f_2)_{\pi z}$ is generated by $\pi z - 1$ and $\varphi\pi T_i$, and $\psi^{-1}(1 \otimes (\pi z - 1)) = z - 1$, by 6.12(iv), and

$$\psi^{-1}(1 \otimes \varphi\pi T_i) = \sum_{j=0}^{\infty} (-y/z)^j \varphi \operatorname{ad} x^j T_i / j! = \varphi T_i + A(z - 1)$$

by 6.12(iv) and 6.1. φ commutes with ad x since φ is a g-module map. Since ψ is an algbra isomorphism (6.12),

$$I(f) = \langle z - 1, \varphi T_2, \ldots, \varphi T_r\rangle = \langle z - 1, \varphi\Gamma S_2, \ldots, \varphi\Gamma S_r\rangle,$$

by 6.8.

8.7. This completes the proof of Theorem 7.1.

9. A counterexample to a claim of Kirillov. In [6] Kirillov stated that for a nilpotent Lie algebra g of dimension n and an $f \in g^*$ there is a basis $\{x_1, \ldots, x_n\}$ for g such that the orbit through f is described by polynomial equations

$$x_i = f_i(x_{k+1}, \dots, x_{2k}), \qquad i = 2k+1, \dots, n.$$

7.1(3) shows us that we may describe the orbit as

$$h_i = R_i(x_2, \dots, x_k, y_1, \dots, y_k),$$

where the variables actually occurring in R_i have order $< o(h_i)$, and where $\{h_1, \dots, h_r\}$ and $\{h_1, \dots, h_r, x_1, \dots, x_k, y_1, \dots, y_k\}$ are bases for g^f and g, respectively. Thus it would be natural to attempt to improve 7.1(3) by reducing to k the number of variables needed in R_i. The example following shows that this cannot be done in general, that Kirillov's claim is too strong. We shall describe the computation in detail, to illustrate the procedure outlined in 7.4.

Let g be the six-dimensional Lie algebra with basis $\{u_1, \dots, u_6\}$ and nonzero products determined by $[u_1,u_2] = u_3$, $[u_1,u_3] = u_4$, $[u_1,u_4] = u_5$, $[u_2,u_5] = u_6$, $[u_3,u_4] = -u_6$. This algebra is g_6^{18} in Vergne [9]. Let the filtration be given by the ascending central series, so that we may let $u_1, \dots, u_6$ have orders 5,5,4,3,2,1, respectively. Let $f(u_6) = 1, f(u_i) = 0$ for $i \neq 6$.

We shall choose a filtered basis $\{z,h,x_1,x_2,y_1,y_2\}$ as in 7.4. f is irreducible in g. In the notation of 2.2 let $z = u_6$, $y_1 = u_5$, $x_1 = u_2$, $B = \text{kernel ad}\, y_1 = \langle u_1, u_3, u_4, y_1, z\rangle$. Let $C = B/\langle y_1\rangle = \langle \bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{z}\rangle$, with $\bar{u} = \pi u$. Let f_1 and f_2 be the induced functionals on B and C, respectively. f_2 is irreducible in C and we apply 2.2 again, choosing $\bar{y}_2 = \bar{u}_4$, $\bar{x}_2 = -\bar{u}_3$. We lift $\bar{y}_2$ and $\bar{x}_2$ to g, preserving the order, by choosing $y_2 = u_4$ and $x_2 = -u_3$ in g.

Let $B_2 = \text{kernel ad}\, \bar{y}_2 = \langle \bar{y}_2, \bar{u}_1, \bar{z}\rangle$. B_2 is abelian, so, letting f_3 be the restriction of f_2 to B_2, $J(f_3) = \langle \bar{y}_2, \bar{u}_1, \bar{z} - 1\rangle$. Let

$$\Gamma_i = \sum (-y_i)^j (\text{ad}\, x_i)^j, \quad \text{and} \quad \bar{\Gamma}_i = \sum (-\bar{y}_i)^j (\text{ad}\, \bar{x}_i)^j, \qquad i = 1,2.$$

By 7.3, $J(f_2) = \langle \bar{\Gamma}_2(\bar{u}_1), \bar{\Gamma}_2(\bar{z} - 1)\rangle$ and, thus, by 7.2 and 6.5, $J(f_1) = \langle u_1 - 2u_4^2, u_6 - 1\rangle$. By 7.3,

$$\begin{aligned} J(f) &= \left\langle \Gamma_1(u_1 - 2u_4^2), u_6 - 1\right\rangle \\ &= \left\langle u_1 - 2u_4^2 + u_5u_3, u_6 - 1\right\rangle \\ &= \langle \Gamma_1\Gamma_2(u_1) - f(u_1), \Gamma_1\Gamma_2(u_6) - f(u_6)\rangle \end{aligned}$$

as in 7.4. We observe that the equations defining the orbit through f are $u_1 = 2u_4^2 - u_5u_3 = R(x_2,y_1,y_2)$ and $u_6 = 1$ and that R cannot be expressed as a polynomial in fewer variables. Thus the claim of Kirillov is not verified for this example.

10. **Example:** $\varphi J(f) \neq I(f)$. The following example of Nicole Conze (kindly supplied by Michèle Vergne) shows that $\varphi J(f) \neq I(f)$ in general. Let $g = \langle u_1, u_2, u_3, u_4\rangle$, with $[u_1, u_2] = u_3$, $[u_1, u_3] = u_4$. Let $f(u_4) = 1, f(u_i) = 0$ for $i \neq 4$. Then f is irreducible and taking $x = u_1, y = u_3, z = u_4$, we get $J(f_1) =$

$\langle u_2, u_3, u_4 - 1\rangle_{S(B)}$ and, thus, $J(f) = \langle 2u_2 - u_3^2, u_4 - 1\rangle_{S(g)}$ and $I(f) = \langle 2u_2 - u_3^2, u_4 - 1\rangle_{U(g)}$, by 7.3. Let $P = u_1^2(2u_2u_4 - u_3^2)$. Then P is in $J(f)$ but φP equals $\frac{1}{6}$ mod $I(f)$.

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